Partial fraction 1

1. Partial fraction: $\frac{9x-2}{x^2-x-6}$

$$\frac{9x-2}{x^2-x-6}=\frac{9x-2}{(x-3)(x+2)}=\frac{\left[\frac{9x-2}{x+2}\right]_{x=3}}{x-3}+\frac{\left[\frac{9x-2}{x-3}\right]_{x=-2}}{x+2}=\frac{\left[\frac{9(3)-2}{3+2}\right]}{x-3}+\frac{\frac{9(-2)-2}{(-2)-3}}{x+2}=\frac{\mathbf{5}}{\boldsymbol{x-3}}+\frac{\mathbf{4}}{\boldsymbol{x+2}}$$

2. Partial fraction: $\frac{3x}{x^3-1}$

(1) $x^3-1=(x-1)(x^2+x+1)$

(2) $(x-1)^2=(x^2+x+1)-3x$

$3x=(x^2+x+1)-(x-1)^2$

(3) $\frac{3x}{x^3-1}=\frac{(x^2+x+1)-(x-1)^2}{(x-1)(x^2+x+1)}=\frac{\mathbf{1}}{\boldsymbol{x-1}}-\frac{\boldsymbol{x-1}}{\boldsymbol{x^2+x+1}}$

3. Partial fraction: $\frac{x^2+1}{(x-1)^2(x^2+2x+2)}$

Let $\frac{x^2+1}{(x-1)^2(x^2+2x+2)}=\frac{A}{(x-1)^2}+\frac{f(x)}{(x-1)(x^2+2x+2)}$

Then $A=\left[\frac{x^2+1}{x^2+2x+2}\right]_{\mathrm{x}=1}=\frac{1^2+1}{1^2+2(1)+2}=\frac{2}{5}$

Hence $\frac{x^2+1}{(x-1)^2(x^2+2x+2)}=\frac{\frac{2}{5}}{(x-1)^2}+\frac{f(x)}{(x-1)(x^2+2x+2)}$

$\therefore (x-1)f(x)=(x^2+1)-\frac{2}{5}(x^2+2x+2)=\frac{1}{5}(x-1)(3x-1)$

$\therefore f(x)=\frac{1}{5}(3x-1)$

Hence $\frac{x^2+1}{(x-1)^2(x^2+2x+2)}=\frac{\frac{2}{5}}{(x-1)^2}+\frac{1}{5}\frac{(3x-1)}{(x-1)(x^2+2x+2)}$ ... (1)

Consider $\frac{3x-1}{(x-1)(x^2+2x+2)}=\frac{B}{x-1}+\frac{g(x)}{x^2+2x+2}$

Then $B=\left[\frac{3x-1}{x^2+2x+2}\right]_{x=1}=\frac{3-1}{1+2+2}=\frac{2}{5}$

$$\frac{3x-1}{(x-1)(x^2+2x+2)}=\frac{\frac{2}{5}}{x-1}+\frac{g(x)}{x^2+2x+2}$$

$(x-1)g(x)=3x-1-\frac{2}{5}(x^2+2x+2)$

$g(x) = -\frac{1}{5}(2x-9)$

$\frac{3x-1}{(x-1)(x^2+2x+2)} = \frac{\frac{2}{5}}{x-1} + \frac{-\frac{1}{5}(2x-9)}{x^2+2x+2} \dots (2)$

$(2) \downarrow (1), \frac{x^2+1}{(x-1)^2(x^2+2x+2)} = \frac{\frac{2}{5}}{(x-1)^2} + \frac{\frac{2}{25}}{x-1} + \frac{-\frac{1}{25}(2x-9)}{x^2+2x+2}$

4. Partial fraction: $\frac{5x^2+x+6}{(3-2x)(x^2+4)}$

Let $\frac{5x^2+x+6}{(3-2x)(x^2+4)} = \frac{A}{3-2x} + \frac{f(x)}{x^2+4}$

Then $A = \left[\frac{5x^2+x+6}{x^2+4}\right]_{x=\frac{3}{2}} = \frac{5\left(\frac{3}{2}\right)^2+\left(\frac{3}{2}\right)+6}{\left(\frac{3}{2}\right)^2+4} = 3$

Hence $\frac{5x^2+x+6}{(3-2x)(x^2+4)} = \frac{3}{3-2x} + \frac{f(x)}{x^2+4} = \frac{3(x^2+4)+(3-2x)f(x)}{(3-2x)(x^2+4)}$

$f(x) = \frac{5x^2+x+6-3(x^2+4)}{3-2x} = \frac{2x^2+x-6}{3-2x} = \frac{(2x-3)(x+2)}{3-2x} = -(x+2)$

$\therefore \frac{5x^2+x+6}{(3-2x)(x^2+4)} = \frac{3}{3-2x} - \frac{x+2}{x^2+4}$

5. Partial fraction: $\frac{x+1}{(x^2+1)(x^2+x+1)}$

The two factors in the denominator are quite close.

However, the numerator $x+1$ cannot be rewriten as $(x^2+x+1)-(x^2+1)$.

Now, $(x-1)(x^2+x+1) = x^3-1$ is familiar.

We don't like the x^3 term.

Working on another factor, $x(x^2+1) = x^3+x$

Hence $x(x^2+1)-(x-1)(x^2+x+1) = x+1$. Bingo!

$\therefore \frac{x+1}{(x^2+1)(x^2+x+1)} = \frac{x(x^2+1)-(x-1)(x^2+x+1)}{(x^2+1)(x^2+x+1)} = \frac{x}{x^2+x+1} - \frac{x-1}{x^2+x}$

6. Partial fractions $\frac{7x^2-10x+10}{(x-1)^3}$

$\frac{7x^2-10x+10}{(x-1)^3} = \frac{7[(x-1)+1]^2-10[(x-1)+1]+10}{(x-1)^3} = \frac{7[(x-1)^2+2(x-1)+1]-10[(x-1)+1]+10}{(x-1)^3}$

$$=\frac{7(x-1)^2+4(x-1)+7}{(x-1)^3}=\frac{\mathbf{7}}{\boldsymbol{x-1}}+\frac{\mathbf{4}}{\boldsymbol{(x-1)^2}}+\frac{\mathbf{7}}{\boldsymbol{(x-1)^3}}$$

7. Partial fraction: $\frac{9x^2+34x+14}{(x+2)(x^2-x-12)}$

$$\frac{9x^2+34x+14}{(x+2)(x^2-x-12)}=\frac{9x^2+34x+14}{(x+2)(x+3)(x-4)}$$

$$=\frac{\left[\frac{9x^2+34x+14}{(x+3)(x-4)}\right]_{x=-2}}{x+2}+\frac{\left[\frac{9x^2+34x+14}{(x+2)(x-4)}\right]_{x=-3}}{x+3}+\frac{\left[\frac{9x^2+34x+14}{(x+2)(x+3)}\right]_{x=4}}{x-4}$$

$$=\frac{\mathbf{3}}{\boldsymbol{x+2}}-\frac{\mathbf{1}}{\boldsymbol{x+3}}+\frac{\mathbf{7}}{\boldsymbol{x-4}}$$

8. Partial fraction: $\frac{x-1}{(3x-1)(2x+5)}$

$$\frac{x-1}{(3x-1)(2x+5)}=\frac{\left[\frac{x-1}{2x+5}\right]_{x=\frac{1}{3}}}{3x-1}+\frac{\left[\frac{x-1}{3x-1}\right]_{x=-\frac{5}{2}}}{2x+5}=\frac{\left[\frac{\frac{1}{3}-1}{2\left(\frac{1}{3}\right)+5}\right]}{3x-1}+\frac{\left[\frac{-\frac{5}{2}-1}{3\left(-\frac{5}{2}\right)-1}\right]}{2x+5}=\frac{-\frac{\mathbf{2}}{\mathbf{17}}}{\boldsymbol{3x-1}}+\frac{\frac{\mathbf{7}}{\mathbf{17}}}{\boldsymbol{2x+5}}$$

9. Partial fraction: $\frac{x^4}{(x^2-x+1)(x^2+2)^2}$

(1) $(x^2+x+1)(x^2-x+1)=(x^2+1)^2-x^2=x^4+x^2+1$

$(x^2+2)^2=x^4+4x^2+4$

(2) Let $A(x^4+x^2+1)+B(x^4+4x^2+4)=x^4 \Longrightarrow \begin{cases} A+B=1 \\ A+4B=0 \end{cases} \Longrightarrow A=\frac{4}{3}, B=-\frac{1}{3}$

(3) $\frac{x^4}{(x^2-x+1)(x^2+2)^2}=\frac{\frac{4}{3}(x^2+x+1)(x^2-x+1)-\frac{1}{3}(x^2+2)^2}{(x^2-x+1)(x^2+2)^2}=\frac{4}{3}\left[\frac{x^2+x+1}{(x^2+2)^2}\right]-\frac{1}{3}\left[\frac{1}{x^2-x+1}\right]$

$$=\frac{4}{3}\left[\frac{(x^2+2)+(x-1)}{(x^2+2)^2}\right]-\frac{1}{3}\left[\frac{1}{x^2-x+1}\right]=\frac{\mathbf{4}}{\mathbf{3}}\left[\frac{\mathbf{1}}{\boldsymbol{x^2+2}}\right]+\frac{\mathbf{4}}{\mathbf{3}}\left[\frac{\boldsymbol{x-1}}{\boldsymbol{(x^2+2)^2}}\right]-\frac{\mathbf{1}}{\mathbf{3}}\left[\frac{\mathbf{1}}{\boldsymbol{x^2-x+1}}\right]$$

10. Partial fraction: $\frac{3x-2}{(x+1)^2(x-3)^2}$

(1) $\frac{3x-2}{(x+1)(x-3)}=\frac{\left[\frac{3x-2}{x-3}\right]_{x=-1}}{x+1}+\frac{\left[\frac{3x-2}{x+1}\right]_{x=3}}{x-3}=\frac{5}{4}\frac{1}{x+1}+\frac{7}{4}\frac{1}{x-3}$ ... (1)

(2) $\frac{1}{(x+1)(x-3)}=\frac{\left[\frac{1}{x-3}\right]_{x=-1}}{x+1}+\frac{\left[\frac{1}{x+1}\right]_{x=3}}{x-3}=-\frac{1}{4}\frac{1}{x+1}+\frac{1}{4}\frac{1}{x-3}$ ... (2)

(3) $(1)\times(2)$, $\frac{3x-2}{(x+1)^2(x-3)^2}=-\frac{5}{16}\frac{1}{(x+1)^2}-\frac{1}{8}\frac{1}{(x+1)(x-3)}+\frac{7}{16}\frac{1}{(x-3)^2}$

$$=-\frac{5}{16}\frac{1}{(x+1)^2}-\frac{1}{8}\left(-\frac{1}{4}\frac{1}{x+1}+\frac{1}{4}\frac{1}{x-3}\right)+\frac{7}{16}\frac{1}{(x-3)^2}\text{ , by (2)}$$

$$=-\boldsymbol{\frac{5}{16}\frac{1}{(x+1)^2}+\frac{1}{32}\frac{1}{x+1}-\frac{1}{32}\frac{1}{x-3}+\frac{7}{16}\frac{1}{(x-3)^2}}$$

Yue Kwok Choy

1-10-2021